A Logic for Iterated Belief Revision

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***Abstract*—Representing belief information is a fundamental problem in the field of belief revision. The AGM framework uses a deductively closed set of formulas, known as a theory, to represent the belief information of an agent, because the belief information of a rational agent should satisfy properties similar to a theory. However, in the iterated revision setting, the DP framework uses conditional beliefs like $(\varphi \mid \psi)$ to represent such information, which is not natural, as conditional beliefs are not formulas and logical connections between them cannot be characterized clearly. In this paper, we propose a novel logic system for representing belief information under iterated revision as a theory in this logic system, which is more natural than the approach of the DP framework. We also prove the soundness and completeness of the logic system, such that it is readily usable for performing iterated revision. Finally, we showed that this logic system is powerful enough to represent general epistemic state as a theory, which is more general than the representation of the DP framework.**

***Index Terms*—belief revision, AGM framework, epistemic state, belief algebra**

I. Introduction

Knowledge representation and reasoning is a core part of artificial intelligence research, and belief revision [1] is an important branch of knowledge representation and reasoning. It mainly concerns how an agent changes her belief when she gets new evidence (i.e. new belief information). The core issues in this field involve the representation of belief information and the formulation of belief revision rules. Logic based belief revision has been studied in the past three decades [2]–[10]. The best-known approach in this field is the AGM framework by Alchourron, Gärdenfors, and Makinson [2], proposing solutions to both of the core issues of belief revision.

The AGM framework characterizes belief revision under some background logic that includes classical propositional logic, and represents an agent's current belief K (called a *belief set*) as a set of formulas that is deductively closed, and represents the new evidence by a single formula φ. The revision result is also represented as a belief set $K \circ \varphi$, where $\circ$ (called revision operator) is a map from belief set and formula to belief set. The AGM framework identified eight postulates as the basic rules of believe revision operator. To semantically characterise AGM operators, researchers introduced several different models such as system of spheres [11], epistemic entrenchment [12], and total preorders on worlds [3]. In [3], Katsuno and Mendelzon considered the situation when the background language is a finite propositional logic language. They showed an important representation theorem that for each belief set K and an AGM revision operator, there is a total preorder $\preceq$ on worlds such that for any formula φ, the revision result $K \circ \varphi$ is totally decided by the minimal φ worlds w.r.t. $\preceq$. This representation theorem means that an AGM revision operator can be completely characterized by a total preorder on worlds, which represents the agent's "preference" on worlds. To syntactically characterize the preference of the agent, Darwiche and Pear (DP for short) introduced the concept of *conditional belief* [4]. A conditional belief has the form $(\varphi \mid \psi)$, which means when the agent believes ψ she will also believes φ. The DP framework uses *epistemic state* instead of belief set to characterize the agent's belief information, where an epistemic state consists of a belief set and some conditional beliefs. The belief revision framework based on epistemic states can better model the iterated belief revision process by maintaining the preference of the agent in subsequent revisions [4], [13], while the traditional AGM paradigm discards such information. Based on the DP framework, iterated belief revision has been extensively studied in the past two decades [5], [8], [9], [14], [15].

However, there are still some fundamental issues that remain unresolved in iterated belief revision, one of which lies in the DP framework. The DP framework uses conditional beliefs like $(\varphi \mid \psi)$ to represent the preference of agent, but $(\varphi \mid \psi)$ is not a "formula" in the underlying logic. This means things like "$\neg(\varphi \mid \psi)$", "$(\varphi_1 \mid \psi_1) \wedge (\varphi_2 \mid \psi_2)$" are not defined,

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which is not natural. Moreover, the relation between $(\varphi \mid \psi)$ and $(\neg\varphi \mid \psi)$ is also not characterized clearly in a syntactic way. In fact, note that in the AGM framework, a rational belief set is equivalent to a deductively closed set of formulas (i.e. a *theory*) under the background logic. It is actually natural to characterize belief information with a theory, because belief information should have the following common-sense properties: if the agent believes φ then she should not believe $\neg\varphi$; if the agent believes φ and ψ then she should also believe $\varphi \wedge \varphi$; if the agent believes φ and φ implies ψ then she should believe ψ. In other words, given a belief set K, for any formula φ, exactly one of the following cases will happen: (1) $\varphi \in K$; (2) $\neg\varphi \in K$; (3) both $\varphi, \neg\varphi \notin K$, where the agent have no preference on φ and $\neg\varphi$, and the agent believes neither φ nor $\neg\varphi$. These are exactly properties of a theory. To resolve the problem of the unnatural representation of belief information in iterated belief revision, we propose a novel logic system to bring belief sets and conditional beliefs together into a unified framework. We will discuss both syntax and semantics to characterize the belief information of an agent. The main contributions of this paper are:

- We proposed a novel logic system for modelling belief information, which is more intuitive in the sense that it is in a similar form as the well-known AGM paradigm;
- We proved the soundness and completeness of the logic system, which means it is readily applicable to performing iterated belief revision based on this logic system;
- We showed the representation power of this logic system, by proving that a general epistemic state, which is more general than epistemic state or belief set, can be represented as a deductively closed set of formulas (a theory) in this system.

The remainder of this paper is structured as follows: Section II introduces basic notions of belief revision, epistemic states, and general epistemic states. Section III proposes and analyses the new logic system for representing belief information. Section IV discusses related work and Section V concludes the paper.

II. Background

A. Notions

In this paper, we restrict our discussion to belief revision in a finite propositional language L, which is built up from a finite set of propositional variables $\mathcal{P}$ and logical connectives such as $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$. We usually use $\varphi, \psi, \dots$ to represent the formula of propositional logic. A *(possible) world* (or *interpretation*) is a mapping from $\mathcal{P}$ to $\{\text{true}, \text{false}\}$, which determines the truth value of propositional variables. Given a world ω and a formula μ in L, we use $\omega \models \mu$ to denote the truth value of μ is true under ω. A *tautology* in L is a formula that is true under any world of L. We denote by $\mathbf{T} = p \vee \neg p$ a fixed tautology in L and $\mathbf{F}$ is an abbreviation for $\neg\mathbf{T}$. We write W for the set of all worlds (interpretations), and denote by $\mathrm{Taut}(L)$ the set of all the tautologies of L. A *theory* K of L is a set of formulas which is deductively closed. More details about propositional logic can be found in [16].

For each formula ψ in L, we denote by $[\psi]$ the set of all worlds of ψ, i.e. $[\psi] = \{\omega \in W \mid \omega \models \psi\}$. For a subset $U \subseteq W$, $\textsc{Form}(U)$ represents a formula whose worlds are exactly those in U. Clearly, we have $\textsc{Form}([\psi]) \equiv \psi$ and $[\textsc{Form}(U)] = U$.

A *(partial) preorder* $\preceq$ on S is a binary relation on S which is reflexive and transitive. A preorder $\preceq$ is called *total* if any two elements in S are comparable under $\preceq$. We write $x \sim y$ if $x \preceq y$ and $y \preceq x$, and $x \prec y$ if $x \preceq y$ but $y \not\preceq x$.

B. Representation in the AGM Framework

In the AGM framework, an agent's belief information is represented as a theory K which is called *belief set*. If the agent is rational, then K is assumed to be consistent. This means that belief information should satisfy the following properties:

(T1) If $\varphi \in K$ then $\neg\varphi$ is not in K.
(T2) If $\varphi_1 \in K$, $\varphi_2 \in K$, then $\varphi_1 \wedge \varphi_2 \in K$.
(T3) If $\varphi_1 \in K$ and $\varphi_1 \rightarrow \varphi_2$ then $\varphi_2 \in K$.

These properties are actually natural for rational human beings. (T1) says an agent should not believe contradictory things; (T2) and (T3) mean that an agent should believe things that can be inferred from known ones. Researchers noted that the representation of belief information in the AGM work has some problems. In fact, there is certain implicit information that was used in the revision but is not maintained in the revision result. This problem makes the AGM framework inappropriate for iterated belief revision (a revision process that may contain a sequence of revisions; see [4]), because in the next round of revision the implicit information was lost.

C. Representation in the DP Framework

After observing that belief set alone is not sufficient to characterize the information needed for iterated belief revision, researchers proposed extra-logical factors such as conditional belief [4], [17], [18] and epistemic entrenchment [12] to overcome this shortcoming.

A *conditional belief* has the form $(\varphi \mid \psi)$, where φ and ψ are formulas in L. An agent has a conditional belief $(\varphi \mid \psi)$ if she will believe φ whenever she believes ψ. Darwiche and Pearl [4] use epistemic states to model an agent's belief states, and propose a framework of iterated belief revision based on epistemic states.

Definition 1. Suppose Ψ consists of a belief set $\mathrm{Bel}(\Psi)$ and a set of conditional beliefs. We say Ψ is an *epistemic state* if the operator $\circ$ defined by Ψ via (EP) satisfies (R^*1)-(R^*6) below.

(EP) $\varphi \in \Psi \circ \psi$ iff $(\varphi \mid \psi) \in \Psi$.
(R^*1) $\Psi \circ \mu$ implies μ.
(R^*2) If $\Psi \wedge \mu$ is satisfiable, then $\Psi \circ \mu \equiv \Psi \wedge \mu$.
(R^*3) If μ is satisfiable, then $\Psi \circ \mu$ is satisfiable.
(R^*4) If $\Psi_1 = \Psi_2$ and $\mu_1 \equiv \mu_2$, then $\Psi_1 \circ \mu_1 \equiv \Psi_2 \circ \mu_2$.
(R^*5) $(\Psi \circ \mu) \wedge \varphi$ implies $\Psi \circ (\mu \wedge \varphi)$.

(R^*6) If $(\Psi \circ \mu) \wedge \varphi$ is satisfiable, then $\Psi \circ (\mu \wedge \varphi)$ implies $(\Psi \circ \mu) \wedge \varphi$.

It was shown in [4] that each epistemic state is equivalent to a total preorder on W, and in this sense, an epistemic state actually depicts the preference (on worlds) of an agent. In fact, if Ψ consists of a belief set $\mathrm{Bel}(\Psi)$ and a set of conditional beliefs, then Ψ is an epistemic state iff there is a total preorder $\preceq$ on worlds such that:

(ES1) $\varphi \in \mathrm{Bel}(\Psi)$ iff there is a world ω in $[\varphi]$ such that $\omega \prec \omega'$ for all $\omega' \in [\neg\varphi]$.
(ES2) $(\beta \mid \alpha) \in \Psi$ iff there is a world ω in $[\alpha \wedge \beta]$ such that $\omega \prec \omega'$ for all $\omega' \in [\alpha \wedge \neg\beta]$.

In the DP framework, iterated belief revision process is equivalent to revising a total preorder $\preceq_\Psi$ by a formula μ, and the revision result is also a total preorder $\preceq_{\Psi\circ\mu}$. This is an extension of the AGM framework, where the revision result of the AGM framework is just a belief set but not an epistemic state (i.e., total preorder). However, the representation of belief information in the DP framework also has some problem, as identified in [19]: it might be too strong for representing belief information in some cases.

D. Representation of belief information by GEP

In the DP framework, conditional beliefs are as important as belief sets, and belief information consists of both a belief set and conditional beliefs. Kern-Isberner [18] noticed that a formula φ can also be seen as a special conditional belief $(\varphi \mid \mathbf{T})$. With this observation, an epistemic state can be seen as a set consisting only of conditional beliefs. Meng et al. [19] proposed to use general epistemic state (GEP) to represent belief information to overcome the problem of epistemic state. A GEP is defined as follows.

Definition 2. Suppose Ψ consists of a belief set $\mathrm{Bel}(\Psi)$ and a set of conditional beliefs. We call Ψ a *general epistemic state* (GEP) if it satisfies (E1)-(E7).

(E1) $\varphi \in \mathrm{Bel}(\Psi)$ iff $(\varphi \mid T) \in \Psi$.
(E2) If $(\beta \mid \alpha) \in \Psi$, then $(\neg\beta \mid \alpha) \notin \Psi$.
(E3) If $(\beta \mid \alpha \vee \beta) \in \Psi$ and $\beta \models \gamma$, then $(\gamma \mid \alpha \vee \gamma) \in \Psi$.
(E4) If $(\beta \mid \alpha \vee \beta) \in \Psi$ and $\gamma \models \alpha$, then $(\beta \mid \gamma \vee \beta) \in \Psi$.
(E5) $(\beta \mid \alpha) \in \Psi$ and $(\gamma \mid \alpha) \in \Psi$ iff $(\beta \wedge \gamma \mid \alpha) \in \Psi$.
(E6) If $\alpha_1 \equiv \alpha_2$ and $\beta_1 \equiv \beta_2$, then $(\beta_1 \mid \alpha_1) \in \Psi$ iff $(\beta_2 \mid \alpha_2) \in \Psi$.
(E7) If φ is consistent, then $(\varphi \mid \varphi) \in \Psi$.

Note that each epistemic state is a GEP, but the opposite is not always true [19]. This representation can characterize more general belief information than the AGM representation as a theory or the DP representation as an epistemic state. Nevertheless, the representation contains conditional beliefs which are not logical formulas, and this makes it hard to characterize the logical relations between conditional beliefs. For instance, $(\varphi \mid \psi)$ and $(\neg\varphi \mid \psi)$ should not be put together into the belief information for a rational agent, and it is natural to have a rule that if the agent holds $(\varphi_1 \mid \psi)$ and $\varphi_1 \rightarrow \varphi_2$, then the agent believes $(\varphi_2 \mid \psi)$. In this paper, we try to build a novel logic system to characterize conditional beliefs directly as formulas, and we expect that it has enough representation power so that both GEP and epistemic state correspond to theories in this logic, while possessing natural properties like the ones in (T1)-(T3).

III. A New Logic for Iterated Belief Revision

In this section, we introduce a novel logic to characterize iterated belief revision, which is based on a finite propositional logic L with propositional variables $\mathcal{P}$.

Recall that if an agent has a conditional belief $(\varphi \mid \psi)$ then she will believe φ whenever she believes ψ. This means that if the agent believes ψ then she believes $\varphi \wedge \psi$, and a conditional belief $(\varphi \mid \psi)$, in some sense, is the same thing as $\varphi \wedge \psi$ is more believable than $\neg\varphi \wedge \psi$. To characterize this kind of preferences, and thus to characterize general epistemic state, we will generalize the concept of formulas by introducing a binary relation on propositional formulas.

Definition 3. Suppose R is a binary relation on L, i.e. $R \subseteq L \times L$, the *relation language* L^R on L is inductively defined as follows:

- If $\langle \varphi, \psi \rangle \in R$, then $P_R(\varphi, \psi)$ is a formula in L^R, and we call the formula with the form $P_R(\varphi, \psi)$ an *R-formula*.
- If $\alpha, \beta \in L^R$, then $\neg\alpha, (\alpha \wedge \beta)$ are formulas in L^R.
- Any formula that is the result of finite applications of the above two rules is a formula in L^R.

Of course, the other standard connectives and constants of L^R like $\vee, \rightarrow, \leftrightarrow, \mathbf{T}, \mathbf{F}$ can be easily defined in terms of $\wedge$ and $\neg$ as in propositional logic. It is then easy to see that $P_R(\varphi, \psi), \neg P_R(\varphi_1, \neg\psi_1 \wedge \psi_2) \rightarrow \neg P_R(\varphi_2, \psi_3)$ are formulas in L^R, however, $\varphi \wedge P_R(R(\varphi_1, \psi_1)$, $P_R(R(\varphi_1, \psi_1), \psi_2)$, $P_R(R(\varphi_1, \psi_1), \varphi_2)$ are not in L^R.

Intuitively, the relation R determines which formulas can be compared, and $P_R(\varphi, \psi)$ represents the statement that φ is preferred over ψ.

Specifically, we can define a relation on L that only consider *pairwise inconsistent* formulas. Suppose $\varphi_1, \varphi_2, \cdots, \varphi_n$ is a set of formulas in L. Then we say $\varphi_1, \varphi_2, \cdots, \varphi_n$ are pairwise inconsistent if for all $i \neq j$ we have $\varphi_i \rightarrow \neg\varphi_j \in \mathrm{Taut}(L)$ (i.e. $[\varphi_i] \cap [\varphi_j] = \emptyset$).

Definition 4. Let B be the binary relation on L s.t. $\langle \varphi, \psi \rangle \in B$ iff $\varphi \rightarrow \neg\psi \in \mathrm{Taut}(L)$. We call L^B the *B-logic* language.

In this case, only the formulas φ and ψ that are inconsistent can be compared, and only then $P_B(\varphi, \psi)$ is a formula in L^B, representing φ is preferred over ψ.

Recall that belief information can be represented by using conditional beliefs in the form of $(\varphi \mid \psi)$, which means $\varphi \wedge \psi$ is more believable than $\neg\varphi \wedge \psi$. Note that $\neg((\varphi \wedge \psi) \wedge (\neg\varphi \wedge \psi)) \equiv \mathbf{T}$, then $\varphi \wedge \psi$ and $\neg\varphi \wedge \psi$ is comparable w.r.t. B and the information $(\varphi \mid \psi)$ can be represented as $P_B(\varphi \wedge \psi, \neg\varphi \wedge \psi)$ in the new language L^B.

In addition to the language itself, the ability to perform inference or reasoning with this language is also critical for it to be used for belief revision. We would inspect the syntax and semantics of B-logic language in the following, in terms of an axiom system. The axiom system $\mathbb{B}$ over L^B contains the following axioms and inference rules:

(PR) All substitution instances of tautologies of propositional logic.
(MP) From α and $\alpha \to \beta$ infer β.
(BP) From $P_B(\varphi, \psi)$ infer $\varphi \to \neg\psi \in \text{Taut}(L)$.
(BR1) $P_B(\varphi, \psi) \to \neg P_B(\psi, \varphi)$.
(BR2) From $\neg\varphi \notin \text{Taut}(L)$ infer $P_B(\varphi, \mathbf{F})$.
(BR3) From $\varphi_1 \to \varphi_2 \in \text{Taut}(L)$ and $\varphi_2 \to \neg\psi_1 \in \text{Taut}(L)$ infer $P_B(\varphi_1, \psi_1) \to P_B(\varphi_2, \psi_1)$.
(BR4) From $\psi_2 \to \psi_1 \in \text{Taut}(L)$ infer $P_B(\varphi_1, \psi_1) \to P_B(\varphi_1, \psi_2)$.
(BR5) From $\varphi \vee \psi \equiv \varphi' \vee \psi'$ infer $P_B(\varphi, \psi) \wedge P_B(\varphi', \psi') \to P_B(\varphi \wedge \varphi', \psi \vee \psi')$.

All of the above correspond to natural assumptions in belief revision. (PR) means that if $\varphi \vee \neg\varphi$ is a tautology of propositional logic, then $\alpha \vee \neg\alpha$ is a axiom in $\mathbb{B}$, where $\alpha \in L^B$. Recall that $P_B(\varphi, \psi)$ means the agent thinks that φ is more believable than ψ. In this situation, (BR1) means that if φ is more believable than ψ, then it is natural that ψ should not be more believable than φ. (BR1) discusses the consistency of belief information like (T1). (BR2) shows if φ is consistent then φ is always more believable than $\mathbf{F}$. For (BR3), if φ_1 implies φ_2 and the agent believes φ_1 is more believable than ψ_1 then she also thinks φ_2 is more believable than ψ_1. Similarly, if φ_1 is more believable than ψ_1 and ψ_2 implies ψ_1, then φ_1 is also more believable than ψ_2 and this is exactly (BR4). (BR3), (BR4) are inspired by (T3). (BR5) is a generalization of the following (BR5*), which is inspired by (T2).

(BR5*) $P_B(\varphi, \neg\varphi) \wedge P_B(\psi, \neg\psi) \to P_B(\varphi \wedge \psi, \neg(\varphi \wedge \psi))$.

(BR5*) means that if the agent believes φ and ψ then she will believe $\varphi \wedge \psi$, i.e., belief information should be closed under conjunction. (BR5) will degenerate to (BR5*) in the case when $\varphi_1 \vee \psi_1 \equiv \varphi_2 \vee \psi_2 \equiv \mathbf{T}$. The reason we want to generalize (BR5*) to (BR5) is that, in GEP we are not only comparing complementary statements like φ and $\neg\varphi$, but also pairs like $\langle \varphi, \psi \rangle \in B$. The next proposition shows that (BR5) is significant in characterizing the reasoning power of belief information, as it can generate natural and necessary inference rules.

Proposition 1. *With the axiom system $\mathbb{B}$, we have the following results:*

(BR6) $(P_B(\varphi_1 \vee \varphi_2, \varphi_3) \wedge P_B(\varphi_1 \vee \varphi_3, \varphi_2)) \to P_B(\varphi_1, \varphi_2 \vee \varphi_3)$.
(BR7) $(P_B(\varphi_1, \psi_1) \wedge P_B(\varphi_1, \psi_2)) \to P_B(\varphi_1, \psi_1 \vee \psi_2)$.

Proof. (1) Take $\varphi = (\varphi_1 \vee \varphi_2), \psi = \varphi_3$ and $\varphi' = \varphi_1, \psi' = (\varphi_2 \vee \varphi_3)$ then we get (BR6) via (BR5). (2) From $P_B(\varphi_1, \psi_1) \wedge P_B(\varphi_1, \psi_2)$ we know $\varphi_1 \to \neg\psi_1 \in \text{Taut}(L)$, $\varphi_1 \to \neg\psi_2 \in \text{Taut}(L)$ by (BP). Since $\varphi_1 \to (\varphi_1 \vee (\psi_2 \wedge \neg\psi_1)) \in \text{Taut}(L)$ and $(\varphi_1 \vee (\psi_2 \wedge \neg\psi_1)) \to \neg\psi_1 \in \text{Taut}(L)$, we have $P_B(\varphi_1, \psi_1) \to P_B(\varphi_1 \vee (\psi_2 \wedge \neg\psi_1), \psi_1)$ by (BR3). In the same way, we have $P_B(\varphi_1 \vee (\psi_1 \wedge \neg\psi_2), \psi_2)$. Take $\varphi = \varphi_1 \vee (\psi_2 \wedge \neg\psi_1), \psi = \psi_1$ and $\varphi' = \varphi_1 \vee (\psi_1 \wedge \neg\psi_2), \psi' = \psi_2$, then we have $P_B(\varphi_1, \psi_1 \vee \psi_2)$ by (BR5). □

(BR6) reveals that if the agent thinks the disjunction of φ_1 and φ_2 is more believable than φ_3, and the disjunction of φ_1 and φ_3 is more believable than φ_2, then the strongest preference of the agent among $\varphi_1, \varphi_2, \varphi_3$ is φ_1. (BR7) shows that if φ_1 is more believable than ψ_1, and φ_1 is more believable than ψ_2, then φ_1 is more believable than the disjunction of ψ_1 and ψ_2. On the basis of (BR6) and (BR7), further propositions on inference rules are drawn as follows.

Proposition 2. *With the axiom system $\mathbb{B}$, we have:*

(BR8) If $\varphi_1 \to \neg\psi_2 \in \text{Taut}(L)$ then $(P_B(\varphi_1, \psi_1) \wedge P_B(\psi_1, \psi_2)) \to P_B(\varphi_1, \psi_2)$.
(BR9) $(P_B(\varphi_1 \vee \varphi_2, \psi_1) \wedge P_B(\varphi_1, \varphi_2)) \to P_B(\varphi_1, \psi_1 \vee \varphi_2)$.

Proof. (1) From $\varphi_1 \to \neg\psi_2 \in \text{Taut}(L)$ and $(P_B(\varphi, \psi_1) \wedge P_B(\psi_1, \psi_2)$, we know $\varphi_1, \psi_1, \psi_2$ are pairwise inconsistent formulas. From $P_B(\varphi_1, \psi_1)$ we have $P_B(\varphi_1 \vee \psi_2, \psi_1)$ by (BR3). Similarly, we have $P_B(\psi_1, \psi_2) \to P_B(\varphi_1 \vee \psi_1, \psi_2)$. Then by (BR7) we have $P_B(\varphi_1, \psi_1 \vee \psi_2)$. Further more we have $P_B(\varphi_1, \psi_2)$ by (BR3) and $\psi_2 \to \psi_1 \vee \psi_2$. (2) From $P_B(\varphi_1 \vee \varphi_2, \psi_1)$, we have $\varphi_1 \vee \varphi_2 \to \neg\psi_1 \in \text{Taut}(L)$. Then we have $P_B(\varphi_1, \varphi_2)) \to P_B(\varphi_1 \vee \psi_1, \varphi_2)$ by (BR3). Following (BR6), we have $P_B(\varphi_1, \varphi_2 \vee \psi_1)$. □

(BR8) shows the preference among formulas has transitivity and (BR9) shows the preference of $\varphi \vee \psi$ is determined by the higher preference portion of them.

Proposition 3. *With the axiom system $\mathbb{B}$, we have the following properties.*
(1) $P_B(\varphi, \psi) \to P_B(\neg\psi, \psi)$.
(2) $P_B(\varphi, \psi_1 \vee \psi_2) \leftrightarrow P_B(\varphi, \psi_1) \wedge P_B(\varphi, \psi_2)$.

Proof. (1) From $P_B(\varphi, \psi)$ we have $\varphi \to \neg\psi$ by (BP). Then we have $P_B(\neg\psi, \psi)$ by (BR3). (2) From $P_B(\varphi, \psi_1 \vee \psi_2)$, we have $\varphi \to \neg(\psi_1 \vee \psi_2) \in \text{Taut}(L)$. This means $\varphi \to \neg\psi_1 \in \text{Taut}(L)$. Then we have $P_B(\varphi, \neg\psi_1)$ by (BR3). Similarly, we have $P_B(\varphi, \neg\psi_1)$, and we have $P_B(\varphi, \psi_1 \vee \psi_2) \to P_B(\varphi, \psi_1) \wedge P_B(\varphi, \psi_2)$. Hence, $P_B(\varphi, \psi_1) \wedge P_B(\varphi, \psi_2) \leftrightarrow P_B(\varphi, \psi_1 \vee \psi_2)$ is straightforward by (BR7). □

Remark 1. The above proposition shows that from $P_B(\varphi, \psi)$ one can induce that the agent believes $\neg\psi$ (or equivalently, $P_B(\psi, \neg\psi)$), and the equivalent characterization between $P_B(\varphi, \psi_1) \wedge P_B(\varphi, \psi_2)$ and $P_B(\varphi, \psi_1 \vee \psi_2)$, which seems like De Morgan law.

Recall that each conditional belief $(\varphi \mid \psi)$ can be represented by a L^B formula $P_B(\varphi \wedge \psi, \neg\varphi \wedge \psi)$. Then each GEP can also be characterized by a set of formulas in L^B, as each GEP consists of conditional beliefs.

Definition 5. Suppose Ψ is a GEP. Then *the set of L^B-formulas* of Ψ, denoted by $\Gamma(\Psi)$, is the set $\{P_B(\varphi \wedge \psi, \neg\varphi \wedge \psi) \mid (\varphi \mid \psi) \in \Psi\}$.

The above definition shows the method of translating a GEP into a set of formulas of L^B. In the next section, we will discuss the semantics of L^B and show the relation between a GEP and a special kind set of formulas, i.e. a *theory* of L^B.

Definition 6. Suppose $\Gamma \subseteq L^B$. Then we call Γ is a *theory* in L^B iff Γ is deductively closed under $\mathbb{B}$. A theory Γ is called *consistent* iff $\mathbf{F} \notin \Gamma$. The *deductive closure* of Γ in L^B, denoted by $\mathrm{Cn}_{\mathbb{B}}(\Gamma)$, is the set of formulas in L^B that is provable by Γ under $\mathbb{B}$, i.e., $\{\alpha \in L^B \mid \Gamma \vdash_{\mathbb{B}} \alpha\}$, where $\Gamma \vdash_{\mathbb{B}} \alpha$ if α is provable by Γ under $\mathbb{B}$.

A. The semantics of L^B

In propositional logic, a truth assignment (i.e., a possible world) gives a truth value of a formula. In modal logic, a modal operator is interpreted by Kripke structure, i.e. a binary relation on W. Similarly, we use belief algebras, which was a concept first introduced in [19], to give the semantics of L^B.

Definition 7. Suppose X is a nonempty set. Then we call $(2^X, \subseteq, \top, \bot, ^-)$ ($(2^X, \subseteq, ^-)$ for short) a *power-set lattice*, where $\subseteq$ is the inclusion relationship between sets, U^- is the complement of U, and $\top = X$ and $\bot = \emptyset$ are the largest element and the smallest element respectively.

It is clear that each power-set lattice is a boolean algebra and we use $a \cap b$ and $a \cup b$ to denote the greatest lower bound and the least upper bound of $\{a, b\}$, respectively.

Definition 8. We call $(G, \subseteq, \gg, \top, \bot, ^-)$ a *belief algebra* if it satisfies

(PL) $(G, \subseteq, ^-)$ is a power-set lattice.

(A0) $\gg$ is a binary relation such that if $a \gg b$ then $a \cap b = \bot$.

(A1) If $a \in G$, then $a \gg \bot$ iff $a \neq \bot$.

(A2) If $a \gg b$, then $b \not\gg a$.

(A3) If $a \subseteq a_1, b_1 \subseteq b$, $a_1 \cap b_1 = \bot$, and $a \gg b$, then $a_1 \gg b_1$.

(A4) If $a = a_1 \cup b_1 = a_2 \cup b_2$ and $a_1 \gg b_1$, $a_2 \gg b_2$, then $a_1 \cap a_2 \gg b_1 \cup b_2$.

(A0) shows that we only need to compare disjoint subset of W. (A1) shows that each nonempty set has a higher preference level than the empty set. (A2) shows $\gg$ is a strict order. (A3) shows that $\gg$ satisfies conditional transitivity. (A4) is inspired by the case that if φ is more believable than $\neg\varphi$ and ψ is more believable than $\neg\psi$ then $\varphi\wedge\psi$ is more believable than $\neg(\varphi\wedge\psi)$.

Definition 9. A *B-model* M is a tuple (G, π), where G is a belief algebra and π is a interpretation function which maps each formula in L to an element of G, such that:

- $\varphi \to \psi \in \mathrm{Taut}(L)$ iff $\pi(\varphi) \subseteq \pi(\psi)$.
- $\pi(\mathbf{T}) = \top, \pi(\neg\varphi) = \pi(\varphi)^-$.
- If $\varphi \not\equiv \psi$ then $\pi(\varphi) \neq \pi(\psi)$.

By translating formulas in L to elements in a belief algebra G, the preference of the agent over formulas becomes the ordering $\gg$ on G. For instance, $\pi(\varphi) \gg \pi(\neg\varphi)$ means that φ is more believable than $\neg\varphi$, and if $\pi(\varphi)$ and $\pi(\psi)$ can not be compared with $\gg$, then the agent has has no idea on which one is more believable.

Definition 10. Suppose G is a belief algebra on 2^W and $M = (G, \pi)$ is a B-model. Then the semantics of L^B can be defined as follows:

- $M \models P_B(\varphi, \psi)$ iff $\pi(\varphi) \gg \pi(\psi)$.
- $M \models \neg P_B(\varphi, \psi)$ iff $M \not\models P_B(\varphi, \psi)$.
- $M \models P_B(\varphi_1, \psi_1) \wedge P_B(\varphi_2, \psi_2)$ iff $M \models P_B(\varphi, \psi)$ and $M \models P_B(\varphi, \psi)$.

For a formula α in L^B, if $M \models \alpha$, then we say M *satisfies* α, or equivalently, α is satisfied by M.

Suppose Γ is a subset of L^B. Then we denote by $M \models \Gamma$ if $\forall \alpha \in \Gamma, M \models \alpha$ for some B-model M, and we denote by $\Gamma \models \beta$ iff for any B-model M, if $M \models \Gamma$ then we have $M \models \beta$. The following theorem shows that any formula in L^B that is provable under $\mathbb{B}$ is satisfied by any B-model.

Theorem 1. *The axiom system $\mathbb{B}$ is sound for L^B with respect to B-models.*

Proof. For (BP), if $M \models P_B(\varphi, \psi)$ then $\pi(\varphi) \gg \pi(\psi)$. Hence, $\pi(\varphi) \cap \pi(\psi) = \bot$ by (A0), and we have $\pi(\varphi) \subseteq \pi(\neg\psi)$ as $\pi(\neg\psi) = \pi(\psi)^-$ is the greatest element of $\{a \mid a \cap \pi(\psi) = \bot\}$ under $\subseteq$. In other word, $\varphi \to \neg\psi \in \mathrm{Taut}(L)$ by the definition of a B-model. Then (BP) is sound. For (BR1), if $M \models P_B(\varphi, \psi)$ then $\pi(\varphi) \gg \pi(\psi)$. By (A2), we have $\pi(\psi) \not\gg \pi(\varphi)$. Then we have $M \models \neg P_B(\psi, \varphi)$ and (BR1) is sound. For (BR2), if $\neg\varphi \notin \mathrm{Taut}(L)$ then $\pi(\varphi) \neq \bot$, and we have $\pi(\varphi) \gg \emptyset$ for any belief algebra G. Hence we have $M \models P_B(\varphi, \mathbf{F})$ and (BR2) is sound. For (BR3), if $\varphi_1 \to \varphi_2$ and $\varphi_2 \to \psi_1 \in \mathrm{Taut}(L)$ then $\pi(\varphi_1) \subseteq \pi(\varphi_2)$ and $\pi(\varphi_2) \cap \pi(\psi_1) = \emptyset$. If $M \models P_B(\varphi_1, \psi_1)$ then $\pi(\varphi_1) \gg \pi(\psi_1)$. By (BR3), we have $\pi(\varphi_2) \gg \pi(\psi_1)$. Then we can conclude that $M \models P_B(\varphi_2, \psi_1)$ and (BR3) is sound. Similarly, (BR4) is sound by (A3) and (BR5) is sound by (A4). $\square$

Next we show that the axiom system $\mathbb{B}$ is also complete, that is, any formula that is satisfied by any B-model is also provable under $\mathbb{B}$.

Theorem 2. *Suppose Γ is a theory in L^B and $\alpha \in L^B$. Then, from $\Gamma \models \alpha$ we have $\Gamma \vdash_{\mathbb{B}} \alpha$.*

Proof. If Γ is inconsistent, then the result is trivial. Therefore we suppose Γ is consistent. By the definition of L^B, the whole proof can be done recursively. We only need to verify the following three cases: $\alpha = P_B(\varphi, \psi)$, $\alpha = \neg P_B(\varphi, \psi)$ and $\alpha = P_B(\varphi, \psi) \wedge P_B(\varphi', \psi')$. To show $\Gamma \vdash_{\mathbb{B}} \alpha$, it is equivalent to show $\alpha \in \mathrm{Cn}_{\mathbb{B}}(\Gamma) = \Gamma$. We will construct a special B-model $M = (G, \pi)$ and show that $M \models \alpha$ is equivalent to $\alpha \in \Gamma$.

Note that $(2^W, \subseteq, ^-)$ is a power-set lattice, where W is the set of all worlds of L. We define a binary relation $\gg$ on 2^W as follows, $U \gg V$ iff $P_B(\text{FORM}(U), \text{FORM}(V)) \in \Gamma$. Next we show $(2^W, \subseteq, \gg, W, \emptyset, -)$ is a belief algebra. By axiom (BP) we know that if $U \gg V$ then $U \cap V = \emptyset$. This means (A0) is satisfied. If $U \neq \emptyset$ then we have $P_B(\text{FORM}(U), \mathbf{F}) \in \Gamma$ by (BR2). This means $U \gg \emptyset$ in G and (A1) is satisfied. If $U \gg V$ then $\neg P_B(\text{FORM}(V), \text{FORM}(U)) \in \Gamma$ by (BR1). Then $P_B(\text{FORM}(V), \text{FORM}(U)) \notin \Gamma$ and $V \not\gg U$. This is equivalent to say that (A2) is satisfied. If $U \subseteq U_1$, $V_1 \subseteq V$, then $\text{FORM}(U) \to \text{FORM}(U_1) \in \text{Taut}(L)$ and $\text{FORM}(V_1) \to \text{FORM}(V) \in \text{Taut}(L)$. If $U_1 \cap V_1 = \emptyset$ then we have $P_B(\text{FORM}(U), \text{FORM}(V)) \to P_B(\text{FORM}(U_1), \text{FORM}(V_1))$ by (BR3) and (BR4). This means if $U \gg V$ then we have $U_1 \gg V_1$, and (A3) is satisfied. If $U = U_1 \cup V_1 = U_2 \cup V_2$ then $\text{FORM}(U_1) \vee \text{FORM}(V_1) \equiv \text{FORM}(U_2) \vee \text{FORM}(V_2)$. Moreover, if $U_1 \gg V_1$ and $U_2 \gg V_2$ then we have $P_B(\text{FORM}(U_1), \text{FORM}(V_1)), P_B(\text{FORM}(U_1), \text{FORM}(V_1)) \in \Gamma$ by the definition of $\gg$. By (BR5), we have $P_B(\text{FORM}(U_1 \cap U_2), \text{FORM}(V_1 \cup V_2)) \in \Gamma$. Thus we have $U_1 \cap U_2 \gg V_1 \cup V_2$ and (A4) is satisfied. Hence $(2^W, \subseteq, \gg, \top, \bot, -)$ is a belief algebra. Furthermore, by the definition of $\gg$, it is easy to verify the following properties:

- $P_B(\varphi, \psi) \in \Gamma$ iff $[\varphi] \gg [\psi]$.
- $\neg P_B(\varphi, \psi) \in \Gamma$ iff $[\varphi] \not\gg [\psi]$.
- $P_B(\varphi, \psi) \wedge P_B(\varphi', \psi') \in \Gamma$ iff $[\varphi] \gg [\psi]$ and $[\varphi'] \gg [\psi']$.

We define $\pi(\varphi) = [\varphi]$ then $M = (G, \pi)$ is a B-model. Moreover, we have $M \models \alpha$ iff $\alpha \in \Gamma$. Notice that, $\alpha \in \Gamma$ is equivalent to $\Gamma \vdash_{\mathbb{B}} \alpha$. Hence, we can conclude that if $\Gamma \models \alpha$ then $\Gamma \vdash_{\mathbb{B}} \alpha$. □

Combining theorems 3 and 4, we can conclude that the axiom system $\mathbb{B}$ is sound and complete for L^B in terms of B-models. From the proof of Theorem 4 it is easy to get the following theorem, which reveals the 1-1 correspondence between a consistent theory in L^B and a belief algebra on 2^W.

Theorem 3. *Suppose Γ is a set of formulas in L^B. Then Γ is a consistent theory in L^B if and only if there is a belief algebra $G = (2^W, \subseteq, \gg, W, \emptyset, -)$, such that $P_B(\varphi, \psi) \in \Gamma$ iff $[\varphi] \gg [\psi]$.*

Note that Meng et al [19] showed that there is also a 1-1 correspondence between GEPs and belief algebras on 2^W via the following lemma.

Lemma 1 ([19]). *Suppose Ψ is a GEP, and $G = (2^W, \subseteq, \gg, W, \emptyset, ^-)$ is a belief algebra.*

- *Given Ψ, we define a binary relation $\gg_\Psi$ on 2^W by $[\alpha \wedge \beta] \gg_\Psi [\alpha \wedge \neg\beta]$ iff $(\beta \mid \alpha) \in \Psi$. Then $(2^W, \subseteq, \gg_\Psi, W, \emptyset, ^-)$ is a belief algebra.*
- *Given G, we define $Gep(G)$ a set of conditional beliefs by $(\beta \mid \alpha) \in Gep(G)$ iff $[\alpha \wedge \beta] \gg [\alpha \wedge \neg\beta]$. Then $Gep(G)$ is a GEP.*
- *There is a 1-1 correspondence between GEPs and belief algebras on 2^W.*

Then, we have the following theorem.

Theorem 4. *Each GEP is equivalent to a theory in L^B.*

Example 1. Suppose L has two propositional variables a and b and $W = \{\omega_1 = a \wedge b, \omega_2 = a \wedge \neg b, \omega_3 = \neg a \wedge b, \omega_4 = \neg a \wedge \neg b\}$, where ω_i can be seen as an interpretation. Take ω_3 for instance: a is false and b is true in ω_3. Let $\varphi_1 = a \wedge b$, $\varphi_2 = \neg a \wedge b$, $\varphi_3 = \neg a \wedge \neg b$. Assume that Bob's current belief information is as follows:

- Bob believes φ_1, or equivalently, Bob thinks φ_1 is more believable than $\neg\varphi_1$.
- Bob thinks φ_2 is more believable than φ_3.

In terms of practical applications, we can think that variable a represents the statement "the thing at a distance is an animal" and variable b represents "the thing at a distance is black"; Bob's current belief information is that "the thing at distance is an animal and is black, and if it is not an animal then it is still black". This belief information can be characterized by a theory $\text{Cn}_{\mathbb{B}}(\Gamma)$, where $\Gamma = \{P_B(\varphi_1, \neg\varphi_1), P_B(\varphi_2, \varphi_3)\}$. The corresponding belief algebra can be generated from $[\varphi_1] \gg [\neg\varphi_1]$ and $[\varphi_2] \gg [\varphi_3]$ with (A0)-(A4). For the sake of brevity, we use $(12, 34)$ to represent $\{\omega_1, \omega_2\} \gg \{\omega_3, \omega_4\}$, and, similarly, $(3, 4)$ for $\{\omega_3\} \gg \{\omega_4\}$. We then have

$$G_1 = \{(12, 34), (12, 3), (12, 4), (123, 4), (124, 3)\\ (3, 4)\} \cup \{(U, \varnothing) \mid U \subseteq W, U \neq \varnothing\}$$

The corresponding GEP is $\Psi = \{(\varphi \mid \psi) \mid [\varphi \wedge \psi] \gg [\neg\varphi \wedge \psi]\}$. This GEP is equivalent to the theory $\text{Cn}_{\mathbb{B}}(\Gamma)$.

Note each epistemic state is also a GEP [19], we have the following corollary.

Corollary 1. *Each epistemic state can be represented as a theory of L^B.*

Example 2. Recall that each epistemic state is equivalent to a total preorder on worlds. Suppose Ψ^* is the epistemic state induced by $\omega_1 \prec \omega_2 \prec \omega_3 \prec \omega_4$ via (ES1) and (ES2). Then it is not difficult to verify that the GEP Ψ in Example 1 is a subset of Ψ^*. The corresponding belief algebra of Ψ^* is

$$\begin{aligned} G_2 = \{&(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 23),\\ &(1, 24), (1, 34), (1, 234), (2, 34), (12, 3), (12, 4),\\ &(12, 34), (13, 2), (13, 4), (13, 24), (14, 2), (14, 3),\\ &(14, 23), (23, 4), (24, 3), (123, 4), (124, 3),\\ &(134, 2)\} \cup \{(U, \varnothing) \mid U \subseteq W, U \neq \varnothing\} \end{aligned}$$

The corresponding theory representation in L^B is $\Gamma^* = \{P_B(\varphi, \psi) \mid [\varphi] \gg [\psi] \text{ holds for } G_2\}$, and Γ^* and Ψ^* are equivalent in characterizing belief information.

IV. Related Work

The representation of belief information is one of the fundamental problems in the knowledge representation and reasoning field. There are different ways of representing beliefs in the literature which fall into two categories: logic-based and measure-based. The former mainly describes belief information with formulas or sets of formulas [2], [11], and the latter with degrees of credibility of information by some measure [20]–[22]. This paper follows the logic-based methods. The AGM framework [2] uses a belief set (i.e., a theory of the background language) to represent belief information. Later, in [3], a belief set under a propositional logic is replaced with an equivalent formula to simplify the representation. In DP framework [4], it was argued that belief sets should be replaced by epistemic states to characterize iterated belief revision, while Meng et al. [19], argue that general epistemic states (GEPs) are more appropriate than epistemic states in describing incomplete belief information. Meanwhile, as a enormous impact of the AGM and DP frameworks, researchers continue working on representing belief information and belief change. Souza et al. [23] consider how well-known postulates from iterated belief revision theory can be characterized by means of belief bases and their counterpart in a dynamic epistemic logic. Qi et al. [24] investigate belief revision in possibilistic logic, which is a weighted logic proposed to deal with incomplete and uncertain information. Gabbay et al. [25] propose a belief revision approach for families of (non-classical) logics whose semantics are first-order axiomatisable. On the other hand, instead of extending the DP framework to more background languages, in this paper, we give a rigorous logical characterization of epistemic states and GEPs by introducing a binary relation on formulas. It is worth noting that Brown et al. [10] define a hierarchy of modal logic that captures logical features of Bayesian belief revision, which is a combination of classical modal logic and Bayesian statistics. However it falls within the measure-based method and is irrelevant to the AGM framework.

V. Conclusion

This paper proposed a novel logic system to represent belief information of the agent. We constructed the language and axioms for the logic system and showed the soundness and completeness of the logic system, which means it is readily usable for performing iterated revision. We illustrated its representation power by proving that a general epistemic state can be naturally represented as a theory in this logic system. In the future, we would like to consider devising iterated revision frameworks based on this logic system.

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